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ON THE SCREW DISLOCATION IN AN INHOMOGENEOUS ELASTIC MEDIUM: THE CASE OF CONTINUOUSLY VARYING ELASTIC MODULI

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Abstract—The elastic fields associated with a screw dislocation at $(x^*, 0)$ in an infinite inhomogeneous elastic medium of an arbitrarily varying shear modulus $\mu(x)$ are obtained by Fourier analysis for the case in which $\mu(x)$ and its derivatives are everywhere continuous and bounded. Using the elastic energy-momentum tensor, the self glide force on the dislocation is calculated. The resulting expression for self-force contains a term $-(b^2/4\pi)(\mathrm{d}\mu/\mathrm{d}x)_{x^*} \ln 1/|\kappa|\varepsilon$, where $\kappa = (1/\mu(x^*))(\mathrm{d}\mu/\mathrm{d}x)_{x^*}$ and ε is the dislocation core radius. Thus κ may be viewed as a reciprocal "characteristic length" which is introduced by the inhomogeneity. Also included is an extension of the results to include the effects of a free surface at $x = 0$.

1. INTRODUCTION

VARIOUS solutions exist in the literature for the states of stress and strain associated with isolated straight dislocation lines in composite media whose elastic constants are only piecewise continuous, i.e. the elastic moduli suffer abrupt discontinuities at internal surfaces, but are otherwise continuous. These solutions have been reviewed in detail by Dundurs [1]. In general this type of inhomogeneity induces image forces which either attract the dislocation toward or repel it away from the surface of modulus discontinuity; the image forces become infinite as the dislocation approaches the surface of discontinuity.

By appropriately linearly superimposing the relevant solutions for isolated dislocations, detailed solutions for the elastic stress states about cracks in such composite materials have been obtained [2–5]. When the leading edge of the crack just reaches the surface of modulus discontinuity, the crack tip stress singularity changes discontinuously [2, 3, 6, 7]; this result severely limits the utility of these continuum models for formulating criteria for crack initiation and propagation in composite or multi-phase materials.

Of course, real materials do not exhibit abrupt discontinuities in elastic constants; the interfaces between adjacent phases of multi-component media are diffuse, so that, within the limits of approximation afforded by the continuum theory of elasticity, the elastic moduli of such media will vary continuously across the diffuse zone separating adjacent phases. Continuously varying elastic constants would exist in the presence of continuous compositional variations in a composite medium. Thus, one might characterize an alloy which has undergone spinodal decomposition as a material whose elastic moduli exhibit periodic fluctuations about some uniform value. Similarly, an alloy containing a random dispersion of precipitates or inclusions could be modelled by considering that the elastic moduli exhibit more or less random fluctuations.

The elastic fields of straight dislocations in media whose moduli are continuous functions of position have not been well-investigated. Therefore, we shall examine the simplest of such problems which one could pose, namely to solve for the elastic state of internal stress due to a screw dislocation in a locally isotropic medium whose shear modulus varies continuously with respect to *one* spatial coordinate which is perpendicular to the dislocation line. For present purposes we further restrict the derivatives of the shear modulus to be everywhere continuous and bounded; the restriction on derivative continuity may be easily removed by a straightforward extension of the technique to be presented [8]. The displacement and stress fields of the screw may be constructed in a rather simple fashion by Fourier analysis, and the result is extended to incorporate the case of a screw dislocation situated in a semi-infinite inhomogeneous medium.

Unlike the situation for an abrupt modulus discontinuity, the self-force on the dislocation in our inhomogeneous medium is due to (1) "image-type forces" which are everywhere finite and (2) a term which is dependent, though rather insensitively, on the dimension of the dislocation core. The interaction force between a free surface and the dislocation will be obtained in a straightforward fashion.

2. THE ELASTIC FIELDS

Consider a locally isotropic elastic medium, infinite in extent, whose shear modulus $\mu(x)$ is given by a single function of position on $-\infty < x < \infty$ (Fig. 1). Elastic stability requires that $\mu(x) > 0$ and we further assume that μ and $d^n\mu/dx^n$ are everywhere continuous and bounded. Let a right-hand screw dislocation be situated at $(x^*, 0)$ (Fig. 1); such a dislocated state would be created by cutting the unstrained medium along $y = 0$ from $x = x^*$ to $x \to \infty$, displacing the bottom of the cut relative to the top of the cut by a constant amount, b, in the positive z-direction and then welding the cut surfaces together. Alternatively, one could make a cut from $x = x^*$ to $x \to -\infty$ and displace the top of the cut relative to the bottom by an amount b in the z-direction. b is the magnitude of the Burgers' vector.

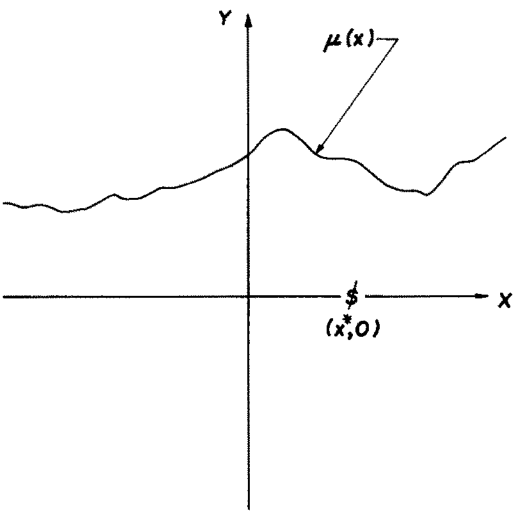

FIG. 1. A screw dislocation ($) at $(x^*, 0)$ in an elastic medium with varying shear modulus $\mu(x)$.

If the state of internal stress is assumed to be one of pure antiplane strain, the only non-vanishing elastic fields are the z-component of displacement (w) and the stresses $\tau_{xz} = \mu(\partial w/\partial x)$ and $\tau_{yz} = \mu(\partial w/\partial y)$. As we are assuming linear small-strain elasticity, when $\mu(x)$ is continuous and no body forces are present, $w(x, y)$ is determined from:

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right)+\mu\frac{\partial^2 w}{\partial y^2} = 0 \quad \text{(elastic equilibrium)} \tag{1}$$

$$\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \quad \text{everywhere continuous and} \rightarrow 0 \text{ as } (x-x^*)^2+y^2 \rightarrow \infty \tag{2}$$

$$\begin{aligned}\lim_{\varepsilon\to 0}\{w(x, -|\varepsilon|)-w(x,|\varepsilon|)\} &= b, \quad x > x^* \\ &= 0, \quad x < x^*.\end{aligned} \tag{3}$$

Writing $w = W(x)\sin\lambda y$, (1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mu\frac{\mathrm{d}W}{\mathrm{d}x}\right)-\lambda^2\mu W = 0. \tag{4}$$

Since $\mu(x) > 0$ and both μ and $\mathrm{d}^n\mu/\mathrm{d}x^n$ are continuous and bounded, (4) has two linearly independent solutions $f_1(x;\lambda)$ and $f_2(x;\lambda)$ which are regular and bounded on $(0,\infty)$ and $(-\infty, 0)$, respectively [9]. It is not difficult to show that

$$\Delta(x) = \frac{\alpha(\lambda)}{\mu(x)}, \tag{5}$$

where Δ is the Wronskian of (f_1, f_2) and $\alpha(\lambda)$ is always positive. Furthermore, for large values of λ, the following asymptotic expansions are valid [9]:

$$f_1(x;\lambda) \sim \frac{\mathrm{e}^{-\lambda x}}{\sqrt{[\mu(x)]}}\left\{1+\sum_{n=1}^{\infty}\left(\frac{1}{2\lambda}\right)^n r_n(x)\right\} \tag{6a}$$

$$f_2(x;\lambda) \sim \frac{\mathrm{e}^{\lambda x}}{\sqrt{[\mu(x)]}}\left\{1+\sum_{n=1}^{\infty}\left(\frac{1}{2\lambda}\right)^n s_n(x)\right\} \tag{6b}$$

$$\alpha(\lambda) \sim 2\lambda\left\{1+\text{order}\left(\frac{1}{\lambda}\right)\right\}. \tag{6c}$$

When the medium is homogeneous, $\mu(x) = \mu_0$ and $f_{1,2} = \mathrm{e}^{\mp\lambda x}/\sqrt{\mu_0}$ with $\alpha(\lambda) = 2\lambda$, provided $\lambda \neq 0$. For $\lambda = 0$, $f_1 = 1$, $f_2 = \int_0^x [\mathrm{d}s/\mu(s)]$ and $\alpha(0) = 1$. A solution of (1) is obtained by an integral superposition over all positive values of λ, i.e.

$$\int_0^{\infty} B(\lambda) f_{1,2}(x;\lambda)\sin\lambda y\,\mathrm{d}\lambda \tag{7}$$

where the choice of f_1 or f_2 is determined by requiring convergence of the integral in accordance with (6a) and (6b). In addition any function $w = \text{const.}$ satisfies (1) and (2).

We now construct the solution for the screw dislocation at $(x^*, 0)$ in the following fashion. For $x < x^*$:

$$w(x, y) = \frac{b}{2\pi}\left\{\pi+2\mu(x^*)\int_0^{\infty}\frac{f_1'(x^*;\lambda)f_2(x;\lambda)}{\alpha(\lambda)}\frac{\sin\lambda y}{\lambda}\,\mathrm{d}\lambda\right\}. \tag{8}$$

For $x > x^*$:

$$w(x, y) = \frac{b}{\pi}\mu(x^*)\int_0^\infty \frac{f_1(x;\lambda)f'_2(x^*;\lambda)}{\alpha(\lambda)}\frac{\sin\lambda y}{\lambda}\,\mathrm{d}\lambda \qquad (y > 0)$$
$$= b+\frac{b}{\pi}\mu(x^*)\int_0^\infty \frac{f_1(x;\lambda)f'_2(x^*;\lambda)}{\alpha(\lambda)}\frac{\sin\lambda y}{\lambda}\,\mathrm{d}\lambda \quad (y < 0). \tag{9}$$

By $f'_1(x^*;\lambda)$ we mean $(\mathrm{d}/\mathrm{d}x^*)f_1(x^*;\lambda)$ or, equivalently, $\{(\mathrm{d}f_1(x;\lambda))/\mathrm{d}x\}_{x=x^*}$. The integrals in (8) and (9) converge, since (6a) and (6b) guarantee that the integrands behave as $\mathrm{e}^{-\lambda|x-x^*|}$ as $\lambda \to \infty$.

The dislocation condition (3) is seen to be trivially satisfied and $(\partial w/\partial x)$, $(\partial w/\partial y)$ are obviously continuous across $y = 0$, so that we need only show that w, $(\partial w/\partial x)$, $(\partial w/\partial y)$ are continuous across $x = x^*$. Equivalently, we may show that the discontinuity in these quantities is zero across x^*. $\partial w/\partial x$ is continuous at $x = x^*$ by virtue of the presence of the common factor $f'_1(x^*;\lambda)f'_2(x^*;\lambda)$ in the integrands for $x > x^*$ and $x < x^*$. Also, since

$$\Delta(x^*) = f_1(x^*;\lambda)f'_2(x^*;\lambda)-f_2(x^*;\lambda)f'_1(x^*;\lambda) = \frac{\alpha(\lambda)}{\mu(x^*)}, \tag{10}$$

for $y \neq 0$

$$\lim_{\varepsilon\to 0}\{w(x^*+\varepsilon, y)-w(x^*-\varepsilon, y)\} = \frac{b}{\pi}\left\{\int_0^\infty \frac{\sin\lambda y}{\lambda}\,\mathrm{d}\lambda-\frac{\pi}{2}\,\mathrm{sgn}(y)\right\}, \tag{11}$$

which vanishes due to the identity

$$\int_0^\infty \frac{\sin\lambda y}{\lambda}\,\mathrm{d}\lambda = \frac{\pi}{2}\,\mathrm{sgn}(y). \tag{12}$$

Furthermore,

$$\lim_{\varepsilon\to 0}\left\{\left(\frac{\partial w}{\partial y}\right)_{x^*+\varepsilon,y}-\left(\frac{\partial w}{\partial y}\right)_{x^*-\varepsilon,y}\right\} = \frac{b}{\pi}\int_0^\infty \cos\lambda y\,\mathrm{d}\lambda$$
$$= b\delta(y), \tag{13}$$

which vanishes except at $y = 0$ [$\delta(y)$ is the Dirac delta function]. Thus (8) and (9) satisfy (1)–(3) except at $(x^*, 0)$ where the fields are singular.

The dislocation stress field is given by:

$x < x^*$:

$$\left.\begin{aligned}\tau_{xz} &= \frac{\mu(x)\mu(x^*)b}{\pi}\int_0^\infty \frac{f'_1(x^*;\lambda)f'_2(x;\lambda)}{\alpha(\lambda)}\frac{\sin\lambda y}{\lambda}\,\mathrm{d}\lambda\\ \tau_{yz} &= \frac{\mu(x)\mu(x^*)b}{\pi}\int_0^\infty \frac{f'_1(x^*;\lambda)f_2(x;\lambda)}{\alpha(\lambda)}\cos\lambda y\,\mathrm{d}\lambda\end{aligned}\right\} \tag{14}$$

$x > x^*$:

$$\left.\begin{aligned}\tau_{xz} &= \frac{\mu(x)\mu(x^*)b}{\pi}\int_0^\infty \frac{f'_1(x;\lambda)f'_2(x^*;\lambda)}{\alpha(\lambda)}\frac{\sin\lambda y}{\lambda}\,\mathrm{d}\lambda\\ \tau_{yz} &= \frac{\mu(x)\mu(x^*)b}{\pi}\int_0^\infty \frac{f_1(x;\lambda)f'_2(x^*;\lambda)}{\alpha(\lambda)}\cos\lambda y\,\mathrm{d}\lambda\end{aligned}\right\}. \tag{15}$$

Thus, the elasticity problem has been formally solved once the function $\mu(x)$ is specified and f_1 and f_2 have been determined. For $\mu(x) = \mu_0$, a constant, the homogeneous solution

$$\left.\begin{aligned} w &= \frac{b}{2\pi}\tan^{-1}\frac{y}{x-x^*} \\ \tau_{xz} &= -\frac{\mu_0 b}{2\pi}\frac{y}{(x-x^*)^2+y^2} \\ \tau_{yz} &= \frac{\mu_0 b}{2\pi}\frac{x-x^*}{(x-x^*)^2+y^2} \end{aligned}\right\} \tag{16}$$

is recovered by virtue of the relations

$$\left.\begin{aligned} \int_0^\infty e^{-\lambda|x-x^*|}\frac{\sin\lambda y}{\lambda}\,d\lambda &= \tan^{-1}\frac{y}{|x-x^*|} \\ \int_0^\infty e^{-\lambda|x-x^*|}\sin\lambda y\,d\lambda &= \frac{y}{(x-x^*)^2+y^2} \\ \int_0^\infty e^{-\lambda|x-x^*|}\cos\lambda y\,d\lambda &= \frac{x-x^*}{(x-x^*)^2+y^2} \end{aligned}\right\} \tag{17}$$

where the principle branch of $\theta = \tan^{-1}(y/|x-x^*|)$ is taken as $-(\pi/2) \le \theta \le \pi/2$.

Using the fact that near $\lambda = 0$

$$\begin{aligned} f_1(x;\lambda) &\sim 1+\text{order}(\lambda) \\ f_2(x;\lambda) &\sim \int_0^x \frac{ds}{\mu(s)}+\text{order}(\lambda), \end{aligned} \tag{18}$$

the results of Erdelyi [9] and Lighthill [10] show that, for fixed x, as $y \to \infty$, the asymptotic forms of (14) and (15) are

$$\tau_{xz} \sim \frac{1}{y}, \qquad \tau_{yz} \sim \frac{1}{y^2}, \tag{19}$$

just as in a homogeneous medium. Roughly speaking, the stresses about a screw dislocation in an inhomogeneous medium decay inversely with distance from $(x^*, 0)$ as $(x-x^*)^2+y^2 \to \infty$.

3. THE CASE OF A FREE SURFACE

When the screw dislocation lies at $(x^*, 0)$ in a semi-infinite medium, $(x, x^* > 0)$, one must require that the surface $x = 0$ be traction-free, i.e. $\mu(0)(\partial w/\partial x)_{x=0} = 0$. This is accomplished by simply superimposing a solution which nullifies the stress $(\tau_{xz})_{x=0}$ given by the first of equations (14) with x set equal to zero. One may easily verify that the superimposed solution must be:

$$\left.\begin{aligned} w^{im} &= -\frac{b}{\pi}\int_0^\infty \frac{f_1(x;\lambda)}{f'_1(0;\lambda)}\,\frac{f'_1(x^*;\lambda)f'_2(0;\lambda)}{\alpha(\lambda)}\,\frac{\sin\lambda y}{\lambda}\,\mathrm{d}\lambda \\ \tau_{xz}^{im} &= -\frac{\mu(x)\mu(x^*)b}{\pi}\int_0^\infty \frac{f'_1(x;\lambda)}{f'_1(0;\lambda)}\,\frac{f'_1(x^*;\lambda)f'_2(0;\lambda)}{\alpha(\lambda)}\,\frac{\sin\lambda y}{\lambda}\,\mathrm{d}\lambda \\ \tau_{yz}^{im} &= -\frac{\mu(x)\mu(x^*)b}{\pi}\int_0^\infty \frac{f_1(x;\lambda)}{f'_1(0;\lambda)}\,\frac{f'_1(x^*;\lambda)f'_2(0;\lambda)}{\alpha(\lambda)}\cos\lambda y\,\mathrm{d}\lambda \end{aligned}\right\} \tag{20}$$

$f'_1(0;\lambda)$ means $\{(\mathrm{d}/\mathrm{d}x)f_1(x;\lambda)\}_{x=0}$. The superscript *im* denotes an "image solution", since, in a certain sense, (20) is an elastic field related to that of a virtual left-hand screw dislocation positioned at $(-x^*, 0)$. As $\lambda \to \infty$ the integrands in (20) behave as $\mathrm{e}^{-\lambda(x+x^*)}$ [see (6)]; this type behavior characterizes a screw dislocation at $(-x^*, 0)$ [equation (17)]. The image solution (20) has no singularities or discontinuities in $x > 0$, so that the elastic fields for the free surface problem are given by those for the infinite medium plus (20).

4. FORCE ON THE DISLOCATION

The self-force in the x-direction per unit length acting upon the dislocation line may be calculated by either (a) using the prescription involving the elastic energy-momentum tensor based upon the classic work of Eshelby [11, 12] or (b) calculating E, the total energy (strain energy in this case) and evaluating $-\partial E/\partial x^*$. The latter technique is more tedious and shall not be dealt with here. Eshelby's energy-momentum tensor prescription shows that the self-force on any elastic singularity or inhomogeneity is determined solely by the fields local to the singularity. In Appendix B we show that the self glide force on the dislocation, F_x, may be expressed as

$$F_x = \frac{b}{2}\{\tau_{yz}(x^*+\varepsilon, 0)+\tau_{yz}(x^*-\varepsilon, 0)\}, \tag{21}$$

where ε is essentially the radius of the dislocation core, a result which has been used previously by Brown [13]. Using equations (14) and (15), (21) becomes

$$F_x = \frac{\mu^2(x^*)b^2}{2\pi}\int_0^\infty \frac{\lim\limits_{\varepsilon\to 0}\{f_1(x^*+\varepsilon;\lambda)f'_2(x^*;\lambda)+f'_1(x^*;\lambda)f_2(x^*-\varepsilon;\lambda)\}}{\alpha(\lambda)}\,\mathrm{d}\lambda. \tag{22}$$

We cannot take the limit $\varepsilon \to 0$ in (22), for then the integral would diverge since

$$\lim_{\lambda\to\infty}\frac{\mathrm{d}}{\mathrm{d}x^*}\{f_1(x^*;\lambda)f_2(x^*;\lambda)\} \sim \frac{1}{\mu(x^*)}+\text{order}\left(\frac{1}{\lambda}\right), \tag{23}$$

so that the integrand in (22) would behave as λ^{-1} as $\lambda \to \infty$. By adding and subtracting the term

$$\mathrm{e}^{-\lambda\varepsilon}\frac{1}{\mu^2(x^*)}\left(\frac{\mathrm{d}\mu}{\mathrm{d}x}\right)_{x^*} = -\frac{\mathrm{d}}{\mathrm{d}x^*}\left(\frac{1}{\mu(x^*)}\right)\mathrm{e}^{-\lambda\varepsilon}$$

to the numerator of the integrand in (22) and then examining the limit as $\varepsilon \to 0$, F_x can be written as

$$F_x = \frac{\mu^2(x^*)b^2}{2\pi}\int_0^\infty \frac{(\mathrm{d}/\mathrm{d}x^*)\{f_1(x^*;\lambda)f_2(x^*;\lambda)-[1/\mu(x^*)]\}}{\alpha(\lambda)}\,\mathrm{d}\lambda$$
$$-\frac{b^2}{2\pi}\left(\frac{\mathrm{d}\mu}{\mathrm{d}x}\right)_{x^*}\lim_{\varepsilon\to 0}\int_0^\infty \frac{\mathrm{e}^{-\lambda\varepsilon}}{\alpha(\lambda)}\,\mathrm{d}\lambda. \tag{24}$$

Now the first integral is independent of ε and converges to a finite value since the integrand is of order λ^{-2} as $\lambda \to \infty$; this term is the counterpart of the "usual image force" one obtains in the case of abrupt modulus discontinuities. The second integral diverges as $\ln\varepsilon$ as $\varepsilon \to 0$, since its behavior for large λ is like that of an exponential integral; at any rate this term depends only upon $(\mathrm{d}\mu/\mathrm{d}x)_{x^*}$, the modulus gradient at the dislocation and ε, the dislocation core radius. An analysis of equation (1) for $(x-x^*)^2+y^2 \leq \varepsilon$ shows that the dominant contribution to the force from the second integral in (24) is (see Appendices)

$$F_x^{\mathrm{core}} = -\frac{b^2}{4\pi}\left(\frac{\mathrm{d}\mu}{\mathrm{d}x}\right)_{x^*}\ln\frac{1}{|\kappa|\varepsilon}, \tag{25}$$

where

$$\kappa = \frac{1}{\mu(x^*)}\left(\frac{\mathrm{d}\mu}{\mathrm{d}x}\right)_{x=x^*}. \tag{26}$$

Implicit in the derivation of (25) is the assumption that

$$|\kappa|\varepsilon \ll 1, \tag{27}$$

which precludes the existence of large fluctuations in $\ln\{\mu(x)/\mu(x^*)\}$ over the dimensions of the dislocation core.

Finally, to terms of order ε and $\varepsilon\ln\varepsilon$, the self-glide force is given by

$$F_x \approx F_x^{\mathrm{core}} + \frac{\mu^2(x^*)b^2}{2\pi}\int_0^\infty \frac{(\mathrm{d}/\mathrm{d}x^*)\{f_1(x^*;\lambda)f_2(x^*;\lambda)-[1/\mu(x^*)]\}}{\alpha(\lambda)}\,\mathrm{d}\lambda \tag{28}$$

F_x^{core} vanishes in a homogeneous medium, as does the remaining term in (28) since $f_1(x^*;\lambda)f_2(x^*;\lambda) \equiv 1/\mu(x^*)$; F_x^{core} is always in the direction of decreasing modulus and may be comparable with the last term in (28) when the screw is situated in a region of large modulus gradient. Using (20) and (21), one notes that the presence of a free surface at $x = 0$ induces an additional or interaction glide force per unit length given by

$$F_x^{\mathrm{int}} = -\frac{\mu^2(x^*)b^2}{\pi}\int_0^\infty \frac{f_1(x^*;\lambda)f_1'(x^*;\lambda)f_2'(0;\lambda)}{f_1'(0;\lambda)\alpha(\lambda)}\,\mathrm{d}\lambda. \tag{29}$$

5. CONCLUSIONS

The present results may be extended to allow for discontinuities in $\mathrm{d}\mu/\mathrm{d}x$ and the screw dislocation near a diffuse grain boundary (approximated by a linear modulus variation) may be studied in detail [8]. Using (14) and (15) as a basis, it should now be possible to study elastic longitudinal shear cracks in inhomogeneous media via the technique of continuously distributed dislocations.

It should be remarked that Wu and Nowinski [14] have studied a particular case of the edge dislocation in an inhomogeneous medium. They were unable to construct a singular Volterra edge (constant Burgers' vector) and dealt instead with a Mann–Somigliana [15] dislocation by allowing stress components which did not contribute to tractions across the slip plane to be multi-valued. As we have shown, when $\mu = \mu(x)$, it is *always* possible to construct a Volterra screw dislocation. This difference may be due to the difference in orientation of the Burgers' vector relative to the direction of modulus variation in the two problems.

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APPENDIX A

Assuming that over the core of the dislocation $\{r^2 = (x-x^*)^2+y^2 \leq \varepsilon^2\}$ the shear modulus $\mu(x)$ is given by

$$\mu(x) = \mu(x^*)\left\{1+\frac{x-x^*}{\mu(x^*)}\left(\frac{d\mu}{dx}\right)_{x=x^*}\right\}, \tag{A1}$$

which is valid provided that ε is such that

$$\frac{\varepsilon}{\mu(x^*)}\left|\left(\frac{d\mu}{dx}\right)_{x=x^*}\right| \ll 1, \tag{A2}$$

equation (1) may be written as

$$\nabla^2 w+\kappa\left\{\frac{\partial w}{\partial r}\cos\theta-\frac{\sin\theta}{r}\frac{\partial w}{\partial\theta}\right\} = 0, \qquad r \leq \varepsilon \tag{A3}$$

where (r, θ) are polar coordinates about $(x^*, 0)$ and

$$\kappa = \frac{1}{\mu(x^*)}\left(\frac{\mathrm{d}\mu}{\mathrm{d}x}\right)_{x=x^*}. \tag{A4}$$

The solution to (A3) satisfying the dislocation condition (3) is

$$w = \frac{b}{2\pi}\{\theta + \tfrac{1}{2}\kappa r \sin\theta \ln|\kappa|r - \tfrac{1}{4}[(\kappa r)^2 \sin 2\theta \ln|\kappa|r]\} + \hat{w}, \tag{A5}$$

where $\hat{w}$ and $\partial\hat{w}/\partial x$, $\partial\hat{w}/\partial y$ are non-singular at $(x^*, 0)$; $\hat{w}$ is similar to the image displacement field in cases in which one allows for an abrupt modulus discontinuity. To terms of order $\kappa r \ln|\kappa|r$, for $r < \varepsilon$, the stresses associated with (A5) are

$$\left.\begin{aligned} \tau_{xz} &= -\frac{\mu(x^*)b}{2\pi}\frac{\sin\theta}{r} + \hat{\tau}_{xz} \\ \tau_{yz} &= \frac{\mu(x^*)b}{2\pi}\left\{\frac{\cos\theta}{r} + \frac{1}{2}\kappa \ln|\kappa|r\right\} + \hat{\tau}_{yz} \end{aligned}\right\} \tag{A6}$$

where $\hat{\tau}_{xz}$ and $\hat{\tau}_{yz}$ are non-singular at $(x^*, 0)$.

APPENDIX B

In the present problem Eshelby's energy-momentum tensor prescription [11, 12] for the self-glide force per unit length on the dislocation reduces to

$$F_x = \varepsilon \int_0^{2\pi} \left\{U \cos\theta - (\tau_{xz}\cos\theta + \tau_{yz}\sin\theta)\frac{\partial w}{\partial x}\right\}_{r=\varepsilon} \mathrm{d}\theta, \tag{B1}$$

where U is the elastic strain energy density, i.e.

$$U = \frac{1}{2\mu(x)}\{\tau_{xz}^2 + \tau_{yz}^2\}. \tag{B2}$$

Strictly speaking (B1) represents the x-component of force on both the dislocations and the inhomogeneities inside the circle $r = \varepsilon$, but as $\varepsilon \to 0$ the amount of inhomogeneity in $r \le \varepsilon$ approaches zero, so that we may consider F_x as a force acting solely upon the dislocation.

Using (A6) in (B1) we can write each field as

$$\left.\begin{aligned} \tau_{xz} &= \tau_{xz}^0 + \hat{\tau}_{xz} \\ \tau_{yz} &= \tau_{yz}^0 + \hat{\tau}_{yz} \\ \frac{\partial w}{\partial x} &= \frac{\partial w^0}{\partial x} + \frac{\partial \hat{w}}{\partial x} \end{aligned}\right\} \tag{B3}$$

and the "$\hat{\ }$-quantities" may be replaced by the average of their values at $(x^* - \varepsilon, 0)$ and $(x^* + \varepsilon, 0)$ and taken outside the integral. Direct integration of the resulting expression for (B1) yields

$$F_x = \frac{b}{2}\{\bar{\tau}_{yz}(x^* + \varepsilon, 0) + \bar{\tau}_{yz}(x^* - \varepsilon, 0)\}, \tag{B4}$$

where

$$\bar{\tau}_{yz}(x,0) = \tau_{yz}(x,0) - \frac{\mu(x)b}{2\pi}\frac{1}{x-x^*}, \tag{B5}$$

i.e. $\bar{\tau}_{yz}$ is that part of the total stress component τ_{yz} which excludes the stress due to the displacement $b\theta/2\pi$. Since

$$\frac{\mu(x^*)b}{2\pi}\left\{\left(\frac{1}{x-x^*}\right)_{x^*+\varepsilon} + \left(\frac{1}{x-x^*}\right)_{x^*-\varepsilon}\right\} = 0, \tag{B6}$$

we may replace $\bar{\tau}_{yz}$ in (B4) by τ_{yz} and obtain equation (21). One notes that (B6) merely restates the well-known result that the self-force on a screw dislocation in an infinite homogeneous medium is zero. Using (A6), excluding the term $\hat{\tau}_{yz}$ which is virtually independent of ε, yields equation (25) for F_x^{core}. The last term in (28) is essentially $\hat{\tau}_{yz}$ evaluated at $(x^*, 0)$.

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Абстракт—С помощью анализа фурье, получаются упругие поля, связанные с винтовой дислокацией для $(x^*, 0)$ в бесконечной, неоднородлой упругой среде, с произвольно изменяющимся моделем сдвига $\mu(x)$, для случая, в котором $\mu(x)$ и его производные являются везде непрерывными и ограниченными. Используя тензор упругой энергии и количества движения, опрелеляется сила самоскольжения. Суммарное выражение для собственной силы имеет член

$$\frac{h^2}{h\pi}\left(\frac{\mathrm{d}x}{\mathrm{d}x}\right)_{x^*} \ln\frac{1}{|k|\epsilon}, \quad \text{rge} \quad k - \frac{1}{\mu(x^*)}\left(\frac{\mathrm{d}x}{\mathrm{d}x}\right)_{x^*}$$

и ϵ обозначает радиус ядра дислокации. Затем, k можно рассматривать как обратную величину "длины характеристики", вызванной неоднородностью. Кроме того, приводится обобщение результатов, с целью учёта эффектов для свободной поверхности для $x = 0$.